# Galactic Dynamics

Problem set 1

Reading: the relevant material is in the Section 3 of Sparke & Gallagher (SG): "Galaxies in the Universe: An Introduction". There are two editions of this book. I have the one from 2000, but you can also use the later one. A presentation of greater depth is in Binney & Tremaine (BT): " Galactic Dynamics", Sections 2.1–2.3. More material can be found in BT sections 2.5, 2.6, 3.1, 3.2 and 3.8, but are not strictly necessary for these problems. There are also some references in the text below and they can also be quite useful.

1. **[10 points]** Circular and escape velocity (general)

   Circular velocity is defined to be a velocity of a test particle (of negligible mass) in a circular orbit at radius $r$:

   $$v_c^2 = r|\mathbf{F}| = r\frac{d\Phi}{dr}$$

   Escape velocity is defined as the velocity at which a star (test particle) can leave the gravitational field $\Phi$ of a system. Assuming that $\Phi(r) \to 0$ when $r \to \infty$:

   $$v_e = \sqrt{2|\Phi(r)|}$$

   What is the meaning of the assumption for escape velocity? Is this a reasonable assumption for a typical stellar system? What are $v_c$ and $v_e$ used to measure?

2. **[15 points]** Circular and escape velocities (applied)

   Derive and plot on the same diagram $v_c$ and $v_e$ as a function of radius for the following set of potentials:

   a) point mass: $\Phi(r) = -\frac{GM}{r}$
   b) homogeneous sphere of size r=a: $\Phi(r) = -\frac{GM(r)}{r}$, where $M(r) = \frac{4}{3}\pi r^3 \rho$, and $\rho$ is density with properties: $\rho(r < a) = const$ and $\rho(r > a) = 0$. Calculate for both $r < a$ and $r > a$.
   c) Plummer model: $\Phi(r) = -\frac{GM}{\sqrt{r^2+b^2}}$

3. **[20 points]** Circular velocity of Power-law models

   Surface brightness profiles of galaxies are often described by power-law functions. These models are natural generalisation of systems where the density drops off as some power with radius: e.g. $\rho(r) = \rho_0(\frac{r_0}{r})^\alpha$. Consider the following density distribution:

   $$\rho_I(r) = \frac{\rho_0}{1 + (r/r_c)^2}$$

   where $\rho_0$ is the central density of the sphere and $r_c$ is the core radius. This density profiles is called *pseudo-isothermal sphere*. Derive the expression for circular velocity and plot it (start from $v_c^2 = \frac{GM(r)}{r}$).

   Show that the density distribution of the Plummer model can be written as:

$$\rho_P(r) = \frac{3b^2}{4\pi}\frac{M}{(r^2+b^2)^{5/2}}$$

Compare the behaviour of the densities and circular velocities of the Plummer model and Pseudo-isothermal sphere. Which one is more representative of real galaxies? Could they describe different components of galaxies?

4. [**20 points**] Circular velocity of two-power-law models

   Two power law models are often used to describe systems which have different behaviour at small and large radii. Their general form is:

$$\rho(r) = \frac{\rho_0}{(r/a)^{\alpha}(1+r/a)^{\beta-\alpha}}$$

   From this general form, four specific model families can be specified:

   a) Dehnen (1993) models, when $\beta = 4$.
   b) Hernquist (1990) models, when $\alpha = 1$ and $\beta = 4$
   c) Jaffe (1983) models, when $\alpha = 2$ and $\beta = 4$
   d) NFW models (Navarro, Frenk &White 1995), when $\alpha = 1$ and $\beta = 3$

   Derive the expression for the mass within radius $r$ and plot the circular velocities for these models on the same diagram. Why are Dehnen models interesting for describing centres of elliptical galaxies (hint: plot projected surface density and discuss)?

5. [**10 points**] Disk models

   Consider axisymmetric potential of the form:

$$\Phi(R,z) = -\frac{GM}{\sqrt{R^2+(a+\sqrt{z^2+b^2})^2}}$$

   with the mass distribution:

$$\rho(R,z) = \left(\frac{b^2M}{4\pi}\right)\frac{aR^2+(a+3\sqrt{z^2+b^2})(a+\sqrt{z^2+b^2})^2}{[R^2+(a+\sqrt{z^2+b^2})^2]^{5/2}(z^2+b^2)^{3/2}}$$

   where $R$ and $z$ are the radial and vertical coordinate, and a and b free parameters. This is a Miyamoto-Nagai model (Miyamoto & Nagai 1975) which can describe a wide range of potentials, from thin disks ($b = 0$) to spheres ($a = 0$, Plummer model).

   Discuss if this potential is a good representation of the rotation curves of spiral galaxies (hint, show what is $v_c$ for large $R$). If one requires that $v_c = const$ at large $R$, what should be the form of the potential $\Phi$?

   Now consider the following potential:

$$\Phi_L(R,z) = \frac{1}{2}v_0^2 ln(R_c^2 + R^2 + \frac{z^2}{q_\Phi^2}) + const$$

   with the density distribution:

$$\rho_L(R,z) = \frac{v_0^2}{4\pi G q_\Phi^2} \frac{(2q_\Phi^2+1)R_c^2 + R^2 + (2-q_\Phi^{-2})z^2}{(R_c^2 + R^2 + z^2 q_\Phi^{-2})^2}$$

where $R$ and $z$ are as before and $q_\Phi$ is the axis ratios of the equipotential surfaces (a measure of flattening of the potential, with $q_\Phi = 1$ for a spherical case), and $v_0$ and $R_c$ are constants. What is the circular velocity in the equatorial plane ($z = 0$) for this potential? This potential is simply referred to as a logarithmic potential.

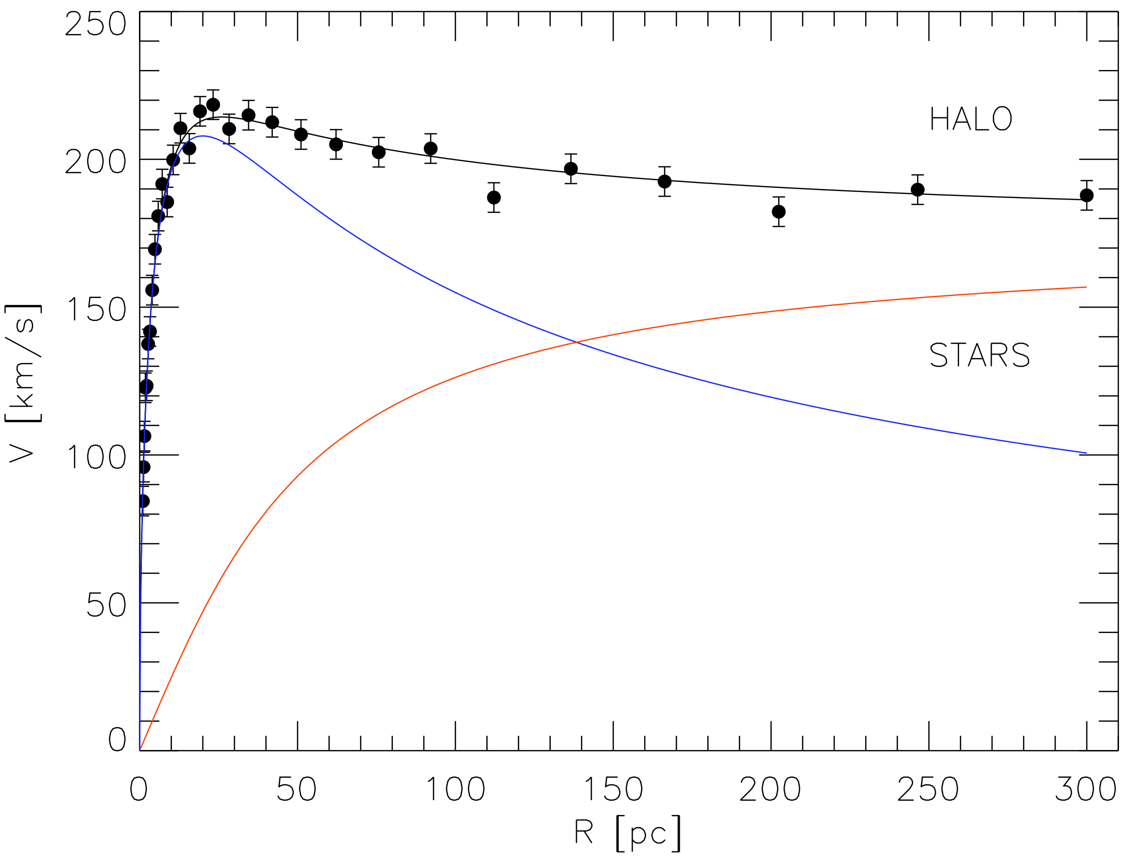


Figure 1: Rotational velocity of a model galaxy made by combining a Hernquist and a pseudo-isothermal model, producing the circular velocity shown with the black line. Blue and red lines show contributions of the individual components, stars and dark matter halo, respectively.

6. **[25 points]** What is the mass of a galaxy?

   Figure 1 shows an "observed"[1] rotational velocity of a very simple galaxy. The galaxy consists of two components, spherical distribution of stars and a spherical dark matter (DM) halo. Their contributions are shown with separate circular velocities (Hernquist for stars and pseudo-isotermal for DM halo).

   - How is the total circular velocity computed? It is obviously not a simple sum of stellar and dark matter contributions.

[1] This is not really true because no seeing effects are taken into account and the errors are the same for all measurements.

- Using the values in the fake_gal.txt file, which contains radius [in parsec] and velocities [in km/s], estimate the total stellar mass and the total dark matter mass within the observed region by constructing a Hernquist and a pseudo-isothermal model for stars and DM, respectively.
- Try to reproduce the data using Jaffe (stars) + logarithmic (DM) and Plummer (stars) + NWF (DM) models. How the total masses of these models compare with Hernquist + pseudo-isothermal masses? Note that you will not be able to perfectly reproduce the data. Just try to get a decent correspondence and estimate their masses by guessing the parameters of the different models. Typically, one doesn't know what is a galaxy made of, and one builds a model and considers what reproduces the data best.

As you will see, some parameters are degenerate, and one can spend a lot of time finding the best fit. Also, note that I didn't give you a light profile of this galaxy, which is something one should use to constrain the stellar part better. We are here interested in order of magnitude results only, and don't spend time making a model that will fit every bump in the data!